Tensor

Syllabus:

Tensor Calculus : Cartesian tensors. Symmetric and antisymmetric tensors. Levi Vivitia tensor density. Pseudo tensors. Dual tensors. Direct product and contraction. Dyads and dyadic. Covariant, Contravariant and mixed tensors. Christoffel symbols and differentiation of tensors.

Q1. What do you mean by a tensor?

Answer: - The tensors are quantities obeying certain transformation laws. The tensor analysis may be regarded as an indispensable part of the study which is rather for the mathematical formulation of natural laws, which remain invariant with respect to different frame of references.

We define a tensor of rank 2n as follows:

$$\overline{A}^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n} = \frac{\partial \overline{x}^{i_1}}{\partial x^{\mu_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{\mu_2}} \ldots\ldots \frac{\partial \overline{x}^{i_n}}{\partial x^{\mu_n}} \frac{\partial x^{\beta_1}}{\partial \overline{x}^{j_1}} \frac{\partial x^{\beta_2}}{\partial \overline{x}^{j_2}} \ldots\ldots \frac{\partial x^{\beta_n}}{\partial \overline{x}^{j_n}} A^{\mu_1 \mu_2 \ldots \mu_n}_{\beta_1 \beta_2 \ldots \beta_n}$$

Tensor is the generalization of the term vector and tensor calculus or rather tensor analysis is a generalization of vector analysis.

Tensors are of important in many areas of physics including general relativity and electromagnetic theory. One of the more prolific sources of tensor quantities is the anisotropic solid. Here the elastic, optical, electrical and magnetic properties may well involve tensors.

Let us consider the flow of current. In this context Ohm's law can be written as .

$$\vec{J} = \sigma \vec{E} \qquad \text{(i)}$$

Where $\vec{J}$ = Current density

$\vec{E}$ = Electric field.

If we have an isotropic medium, σ, the conductivity is a scalar and for the x – component.

$$J_1 = \sigma E_1 \qquad \text{(ii)}$$

However, if the medium is anisotropic, the current density in the x-direction may depend on the electric field in the y- and z-direction as well as on the field in the x-direction. Assuming a linear relationship we can replace equation (ii) by,

$$J_1 = \sigma_{11} E_1 + \sigma_{12} E_2 + \sigma_{13} E_3 \qquad \text{(iii)}$$

In general,

$$J_i = \sum \sigma_{ik} E_k \qquad \text{(iv)}$$

For ordinary 3-dimensional space the scalar conductivity, σ, has given way to a set of nine elements. σ_{ik} to effect the required change in magnitude and direction.

$$\sigma_{ik} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \qquad \text{(v)}$$

This array of 9-elements actually forms a tensor.

Generalizing equation (i), if a relation $\vec{A} = B\vec{C}$ with $\vec{A}$ and $\vec{C}$ as nonparallel vectors holds in all orientations of a Cartesian system, then B is said to be a tensor.

The description of electric, optical and magnetic properties of anisotropic solid (various crystals) may well involve tensors.

Q2. Explain dummy suffix and real suffix with example.

Answer:-

Dummy suffix: When in an indexed expression an index occurs once as a lower index or subscript and again as an upper index or superscript then the index is called a dummy index or dummy suffix.

e.g. In the expression $a_i x^{ij}$, the index 'i' is used as both subscript and superscript and hence it is a dummy index.

Free index or real index: When in an indexed expression an index doesn't occur as a lower index as well as an upper index, then the index is called free index or real index.

e.g. in the expression $a_i x^{ij}$, the index 'i' is a dummy index whereas the index 'j' is a real or free index since it is used as superscript only.

Kronecker delta symbol:

If a particular system of second order is considered and is denoted by δ_j^i : i,j= 1,2,....n and is defined as follows:

$$\delta_j^i = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases}$$

Such a system is called Kronecker delta.

Some properties of Kronecker delta:

(i) If $x^1, x^2, x^3, \ldots, x^n$ are independent variables then,

$$\frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = \delta_j^i$$

(ii) If one deals with n-dimensions, then,

$$\delta_i^i = \delta_j^j = n$$

Proof:

By summation convention;

$\delta_i^i = \delta_1^1 + \delta_2^2 + \ldots + \delta_n^n = 1 + 1 + \ldots\ldots + 1 = n$

(iii) $\delta_j^i \delta_k^j = \delta_k^i$

Proof:

By summation convention;

$\delta_j^i \delta_k^j = \delta_1^i \delta_k^1 + \delta_2^i \delta_k^2 + \ldots\ldots + \delta_i^i \delta_k^i + \ldots + \delta_{n-1}^i \delta_k^{n-1} + \delta_n^i \delta_k^n$

$= 0 + 0 + \ldots\ldots + 1\delta_k^j + \ldots + 0 + 0$

$= \delta_k^j$

(iv) $\delta_k^j = \dfrac{\partial x_i}{\partial x_k} = \dfrac{\partial x_i}{\partial \bar{x}_j}\dfrac{\partial \bar{x}_j}{\partial x_k}$

(v) Kronecker delta is invariant under co-ordinate transformation:

Proof:

$$\bar{\delta}_i^k = \frac{\partial \bar{x}^k}{\partial x^l}\frac{\partial x^m}{\partial \bar{x}^i}\delta_m^l$$

$$= \frac{\partial \bar{x}^k}{\partial x^m}\frac{\partial x^m}{\partial \bar{x}^i}1 \quad [\delta_m^l = 1 \text{ when } l = m \text{ and } 0 \text{ when } l \neq m]$$

$$= \frac{\partial \bar{x}^k}{\partial \bar{x}^i}$$

$$= \delta_i^k$$

(vi) Kronecker delta δ_i^k has the transformation property like that of a mixed tensor of second rank.

Proof:

Let us consider the transformation of a mixed tensor (A_μ^ν) of rank 2.

$$\bar{A}_\mu^\nu = \frac{\partial \bar{x}^\nu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \bar{x}\mu}A_\beta^\alpha \qquad \text{(i)}$$

Kronecker delta is defined as:

$$\delta_l^k = \frac{\partial x^k}{\partial x^l} = \frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^l} \qquad \text{(ii)}$$

Now let us choose, A_β^α to δ_β^α

$\therefore$ From equation (i);

$$\bar{A}_\mu^\nu = \frac{\partial \bar{x}^\nu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \bar{x}\mu}\delta_\beta^\alpha = \frac{\partial \bar{x}^\nu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \bar{x}\mu}\frac{\partial x^\alpha}{\partial x^\beta} = \frac{\partial \bar{x}^\nu}{\partial \bar{x}^\alpha} = \delta_\mu^\nu$$

Now replace $\nu = k$ and $\mu = l$ then,

$$\delta_l^k = \bar{A}_l^k = \frac{\partial \bar{x}^k}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \bar{x}^l}A_\beta^\alpha = \frac{\partial \bar{x}^k}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \bar{x}^l}\delta_\beta^\alpha$$

So from the definition of the mixed tensor, it follows that δ_i^k is a mixed tensor of order 2 with 16 components in 4-dimensional space.

Q. Explain contravariant tensor, covariant and mixed tensor of rank two.

Contravariant tensor: if (n^2) quantities $A^{\alpha\beta}$ (Where α = 1, 2, 3... n; β = 1, 2, 3... n) in a coordinate system are related to (n^2) other quantities $\bar{A}^{\alpha\beta}$ in another coordinate system by the following transformation laws:

$$\bar{A}^{\alpha\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} \frac{\partial \bar{x}^{\beta}}{\partial x^{\nu}} A^{\mu\nu} \qquad \text{(i)}$$

then the quantities $A^{\alpha\beta}$ are said to be the components of a Contravariant tensor of second rank. Equation (i) is the Contravariant law for tensor analysis.

Covariant tensor: if (n^2) quantities $A_{\alpha\beta}$ (Where α = 1, 2, 3... n; β = 1, 2, 3... n) in a coordinate system are related to (n^2) other quantities $\bar{A}_{\alpha\beta}$ in another coordinate system by the following transformation laws:

$$\bar{A}_{\alpha\beta} = \frac{\partial x^{k}}{\partial \bar{x}^{\alpha}} \frac{\partial x^{m}}{\partial \bar{x}^{\beta}} A_{km} \qquad \text{(ii)}$$

then the quantities $A_{\alpha\beta}$ are said to be the components of a Contravariant tensor of second rank. Equation (ii) is the Covariant law for tensor analysis.

Mixed tensor: if (n^2) quantities A^{α}_{β} (Where α = 1, 2, 3... n; β = 1, 2, 3... n) in a coordinate system are related to (n^2) other quantities $\bar{A}^{\alpha}_{\beta}$ in another coordinate system by the following transformation laws:

$$\bar{A}^{\alpha}_{\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{k}} \frac{\partial x^{m}}{\partial \bar{x}^{\beta}} A^{k}_{m} \qquad \text{(iii)}$$

then the quantities A^{α}_{β} are said to be the components of a mixed tensor of second rank. Equation (iii) is the Contravariant law for tensor analysis. An important example of mixed tensor of second rank is Kronecker delta (δ^{α}_{β}).

Q. What is the rank of a tensor?

The rank of a tensor is determined by the number of suffixes or indices attached to it. As a matter of fact the rank of a tensor when raised as a power to the number of dimensions, yields the total number of the components of the tensors and hence the number of the components that represent the tensor. As such a tensor of rank n in four dimensional space has 4^n components. Consequently the rank of a tensor give the number of the mode of changes of the physical quantity when passing from one system to another which is in rotation relative to others.

It is clear from the discussion that a quantity that doesn't change when the axis are rotated is a tensor rank zero. Since the number of mode of changes is then zero. These quantities are named as tensors of rank zero and are scalar. Whereas the tensors of rank one are vectors.

Q. Show that the velocity of a fluid is a Contravariant tensor of rank one.

Assuming that $x_\alpha(t)$ is the coordinate of a moving particle with the time t. Then we have $v^\alpha = \frac{dx_\alpha}{dt}$ as the velocity of the particle. In transformed coordinates the components of velocity are $\bar{v}^\alpha = \frac{d}{dt}\bar{x}_\alpha$.

Now $\bar{v}^\alpha = \frac{d}{dt}\bar{x}_\alpha = \frac{\partial \bar{x}_\alpha}{\partial x_\beta}\frac{dx_\beta}{dt} = \frac{\partial \bar{x}_\alpha}{\partial x_\beta}v^\beta$

This shows that the velocity is a Contravariant tensor of rank one.

Q. Define symmetric and anti-symmetric tensor.

Symmetric tensors:

If a tensor be such that if its two Contravariant or covariant indices are interchanged without altering the value of the tensor, then the tensor is called symmetric tensor with respect to these indices.

Thus if $A_p^{xy} = A_p^{yx}$; then the tensor is said to be symmetric with respect to the indices x and y.

Anti-symmetric tensors:

A tensor is said to be anti-symmetric or skew-symmetric with respect to any to of its Contravariant or covariant indices if the components of the tensor changes sign upon the interchange of the indices.

Thus if $A_p^{xy} = -A_p^{yx}$; then the tensor is said to be anti-symmetric or skew-symmetric with respect to the indices x and y.

Addition of tensors:

The sum of two or more tensors of same rank and type (i.e. same number of covariant and Contravariant indices) is also a tensor of the same rank and type.

Thus if A_x^{lm} and B_y^{pq} are two tensors of rank three then $A_x^{lm} + B_y^{pq} = C_x^{rs}$ is also a tensor of rank three.

Addition of tensors is commutative and associative.

Subtraction of tensors:

The subtraction of two tensors of same rank and type (i.e. same number of covariant and Contravariant indices) is also a tensor of the same rank and type.

Thus if A_x^{lm} and B_y^{pq} are two tensors of rank three then $A_x^{lm} - B_y^{pq} = C_x^{rs}$ is also a tensor of rank three.

Theorem: The sum of two tensors of same rank results in third tensor of same rank.

Proof:

We have to prove $A_{\lambda\mu} + B_{\lambda\mu} = C_{\lambda\mu}$.

Let us consider $A_{\lambda\mu}$ and $B_{\lambda\mu}$ and express them in matrix form as follows:

$$A_{\lambda\mu} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } B_{\lambda\mu} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$\therefore A_{\lambda\mu} + B_{\lambda\mu} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{pmatrix}$$

if the relation between the coefficient a's and b's are such that $a_{\lambda\mu} + b_{\lambda\mu} = c_{\lambda\mu}$

then, $A_{\lambda\mu} + B_{\lambda\mu} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = C_{\lambda\mu}$

which is a tensor of same rank as of $A_{\lambda\mu}$ and $B_{\lambda\mu}$.

Theorem: The difference of two tensors of same rank results in third tensor of same rank.

Proof:

We have to prove $A_{\lambda\mu} - B_{\lambda\mu} = C_{\lambda\mu}$.

Let us consider $A_{\lambda\mu}$ and $B_{\lambda\mu}$ and express them in matrix form as follows:

$$A_{\lambda\mu} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } B_{\lambda\mu} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$\therefore A_{\lambda\mu} - B_{\lambda\mu} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \\ a_{31}-b_{31} & a_{32}-b_{32} & a_{33}-b_{33} \end{pmatrix}$$

if the relation between the coefficient a's and b's are such that $a_{\lambda\mu} - b_{\lambda\mu} = c_{\lambda\mu}$

then, $A_{\lambda\mu} - B_{\lambda\mu} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = C_{\lambda\mu}$

which is a tensor of same rank as of $A_{\lambda\mu}$ and $B_{\lambda\mu}$.

Q. define outer product and show that the outer product of two tensors is a tensor whose order is sum of orders of the two constituent tensors.

The outer product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors.

Thus if r and r' be the ranks of two tensors, then their outer product will be a tensor of rank $r + r'$.

If $A_{\beta}^{\alpha\chi}$ and B_{σ}^{ϕ} are two tensors of rank 3 and 2 respectively, then their outer product $A_{\beta}^{\alpha\chi} B_{\sigma}^{\phi} = C_{\beta\sigma}^{\alpha\chi\phi}$ is a tensor of rank 5 (3 + 2).

Proof:- To prove it, consider the transformation equation of the given tensors as follows:

$$\therefore \bar{A}_{\alpha}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{k}} \frac{\partial \bar{x}^{\nu}}{\partial x^{l}} \frac{\partial x^{m}}{\partial \bar{x}^{\alpha}} A_{m}^{kl} \qquad \text{(i)}$$

$$\text{and } \bar{B}_{\beta}^{\lambda} = \frac{\partial \bar{x}^{\lambda}}{\partial x^{p}} \frac{\partial x^{q}}{\partial \bar{x}^{\beta}} B_{q}^{p} \qquad \text{(ii)}$$

Multiplying (i) and (ii) we get:

$$\bar{A}_{\alpha}^{\mu\nu} \bar{B}_{\beta}^{\lambda} = \frac{\partial \bar{x}^{\mu}}{\partial x^{k}} \frac{\partial \bar{x}^{\nu}}{\partial x^{l}} \frac{\partial x^{m}}{\partial \bar{x}^{\alpha}} \frac{\partial \bar{x}^{\lambda}}{\partial x^{p}} \frac{\partial x^{q}}{\partial \bar{x}^{\beta}} A_{m}^{kl} B_{q}^{p}$$

$$\Rightarrow \bar{C}_{\beta\alpha}^{\mu\nu\lambda} = \frac{\partial \bar{x}^{\mu}}{\partial x^{k}} \frac{\partial \bar{x}^{\nu}}{\partial x^{l}} \frac{\partial \bar{x}^{\lambda}}{\partial x^{p}} \frac{\partial x^{m}}{\partial \bar{x}^{\alpha}} \frac{\partial x^{q}}{\partial \bar{x}^{\beta}} C_{mq}^{klp} \qquad \text{(iii)}$$

The above (equation iii) is a transformation law for a tensor of rank 5. hence the outer product of two tensors one $A_{\alpha}^{\mu\nu}$ of rank 3 and B_{β}^{λ} of rank 2 is a tensor $C_{\beta\alpha}^{\mu\nu\lambda}$ of rank 5.

The outer product of tensors is commutative and associative.

Q. Explain the meaning of contraction with example.

The algebraic operation by which the rank of a tensor may be lowered by 2 (or by any even number) is known as contraction.

In the process of contraction one covariant and one Contravariant index of a mixed tensor are set equal and the repeated index assumed over, the result is a tensor of rank lowered by 2 than the original tensor.

Let us consider a mixed tensor $A^{\mu\nu\sigma}_{\lambda\rho}$ of rank 5 with Contravariant indices μ, ν, σ and co-variant indices λ, ρ.

The transformation low for the given tensor is;

$$\bar{A}^{\mu\nu\sigma}_{\lambda\rho} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\lambda}} \frac{\partial x^{\tau}}{\partial \bar{x}^{\rho}} A^{\alpha\beta\gamma}_{\delta\tau} \qquad \text{(i)}$$

To apply the process of contraction we put $\rho = \sigma$

$$\therefore \bar{A}^{\mu\nu\sigma}_{\lambda\sigma} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\lambda}} \frac{\partial x^{\tau}}{\partial \bar{x}^{\sigma}} A^{\alpha\beta\gamma}_{\delta\tau}$$

Now $\frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \frac{\partial x^{\tau}}{\partial \bar{x}^{\sigma}} = \frac{\partial x^{\tau}}{\partial x^{\gamma}} = \delta^{\tau}_{\gamma} = 1$ when $\tau = \gamma$

$= 0$ when $\tau \neq \gamma$

$$\therefore \bar{A}^{\mu\nu\sigma}_{\lambda\sigma} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \frac{\partial x^{\delta}}{\partial \bar{x}^{\lambda}} A^{\alpha\beta\gamma}_{\delta\gamma} \qquad \text{(ii)}$$

The above is a transformation law for a mixed tensor of rank 3. Hence $A^{\mu\nu\sigma}_{\lambda\sigma}$ is a mixed tensor of rank 3, which may be denoted by $A^{\mu\nu}_{\lambda}$. Thus the process of contraction enables us to obtain a tensor of rank (r-2) from a mixed tensor of rank r.

Q. Explain inner product of two tensors with example.

Inner product: The outer product of two tensors followed by contraction results a new tensor called an inner product of the two tensors and the process is called the inner multiplication of two tensors.

Let us consider two tensors $A^{\mu\nu}_{\sigma}$ and B^{λ}_{ρ}.

The outer product of these two tensors is,

$$A^{\mu\nu}_{\sigma} B^{\lambda}_{\rho} = C^{\mu\nu\lambda}_{\sigma\rho} \qquad \text{(i)}$$

Q. Show that every tensor can be expanded as a sum of symmetric and anti-symmetric tensors with respect to interchange of any two same type of indices.

Proof:

Let $A^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p}$ is a tensor of rank 2p.

$$A^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p} = \frac{1}{2}[A^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p} + A^{i_2 i_1 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p}] + \frac{1}{2}[A^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p} - A^{i_2 i_1 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p}]$$

$$= C^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p} + D^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p}$$

Now

$$C^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p} = \frac{1}{2}[A^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p} + A^{i_2 i_1 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p}]$$

$$= \frac{1}{2}[A^{i_2 i_1 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p} + A^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p}] \text{ {Interchange of } } i_1 \text{ and } i_2\}.$$

$$= C^{i_2 i_1 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots i_p}$$

$\therefore C^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots i_p}$ is symmetric.

And $D^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p} = \frac{1}{2}[A^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p} - A^{i_2 i_1 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p}]$

$$= -\frac{1}{2}[A^{i_2 i_1 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p} - A^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_p}] \text{ {Interchange of } } i_1 \text{ and } i_2\}.$$

$$= -D^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots i_p}$$

$\therefore D^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots i_p}$ is anti-symmetric.

Hence every tensor can be expanded as a sum of symmetric and anti-symmetric tensors with respect to interchange of any two same types of indices.

Q. Show that the symmetry properties of tensors are invariant.

Let us consider a tensor $A_{\lambda\mu\nu}$ be symmetric in its first two indices. That is $A_{\lambda\mu\nu} = A_{\mu\lambda\nu}$. then we have to show that $\bar{A}_{\lambda\mu\nu} = \bar{A}_{\mu\lambda\nu}$.

Now,

$$\bar{A}_{\lambda\mu\nu} = \frac{\partial x^\alpha}{\partial \bar{x}\lambda}\frac{\partial x^\beta}{\partial \bar{x}^\mu}\frac{\partial x^\gamma}{\partial \bar{x}^\nu} A_{\alpha\beta\gamma} \qquad \text{(i)}$$

and

$$\bar{A}_{\mu\lambda\nu} = \frac{\partial x^{\beta}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\alpha}}{\partial \bar{x}\lambda} \frac{\partial x^{\gamma}}{\partial \bar{x}^{\nu}} A_{\beta\alpha\gamma} \qquad \text{(ii)}$$

As the given tensor is symmetric in first two indices, we have,

$$A_{\lambda\mu\nu} = A_{\mu\lambda\nu} \text{ and } A_{\alpha\beta\gamma} = A_{\beta\alpha\gamma}$$

Using this relation and comparing equations (i) and (ii) we find that both the equations are identical.

$\therefore \bar{A}_{\lambda\mu\nu} = \bar{A}_{\mu\lambda\nu}$

Which follows that the given tensor is in other (transformed) coordinate system is also symmetric in first two indices. Hence the symmetric property of a tensor is invariant.

Q. Show that the inner product of two tensors each of rank 1 is a tensor of rank zero.

Let us consider two tensors A^{μ} and B_{ν} each of rank 1.

The outer product of these two tensors is;

$$A^{\mu}B_{\nu} = C^{\mu}_{\nu} \text{ (say)}$$

Now applying the contraction process by setting $\nu = \mu$ we get;

$A^{\mu}B_{\mu} = C^{\mu}_{\mu}$, which is a scalar or tensor of rank zero.

Thus the inner product of two tensors each of rank 1 is a tensor of rank zero.

Q. If $a_{\alpha\beta}x^{\alpha}x^{\beta} = 0$ for all values of variables $x^1, x^2 \ldots, x^n$, then show that $a_{\mu\nu} + a_{\nu\mu} = 0$.

We have,

$$a_{\alpha\beta}x^{\alpha}x^{\beta} = 0 \qquad \text{(i)}$$

Differentiating equation (i) with respect to x^{μ} we get,

$$a_{\alpha\beta}x^{\alpha}\frac{\partial x^{\beta}}{\partial x^{\mu}} + a_{\alpha\beta}x^{\beta}\frac{\partial x^{\alpha}}{\partial x^{\mu}} = 0$$

$$\Rightarrow a_{\alpha\beta}x^{\alpha}\delta^{\beta}_{\mu} + a_{\alpha\beta}x^{\beta}\delta^{\alpha}_{\mu} = 0$$

$$\Rightarrow a_{\alpha\mu}x^{\alpha} + a_{\mu\beta}x^{\beta} = 0 \ldots\ldots\ldots\ldots (ii)$$

Again differentiating equation (ii) with respect to x^{ν} we get;

$$\Rightarrow a_{\alpha\mu}\frac{\partial x^{\alpha}}{\partial x^{\nu}} + a_{\mu\beta}\frac{\partial x^{\beta}}{\partial x^{\nu}} = 0$$

$$\Rightarrow a_{\alpha\beta}\delta^{\alpha}_{\nu} + a_{\alpha\beta}\delta^{\beta}_{\nu} = 0$$

$$\boxed{\Rightarrow a_{\nu\mu} + a_{\mu\nu} = 0}$$

Hence the required result.

Q. Show that $\frac{\partial A_\lambda}{\partial x_\mu}$ is not a tensor although A_λ is a co-variant tensor of rank one.

We have:

$$\bar{A}_\alpha = \frac{\partial x^\lambda}{\partial \bar{x}^\alpha} A_\lambda \text{ (Covariant law)} \qquad \text{(i)}$$

Now differentiating equation (i) with respect to $\bar{x}_\beta$ we get;

$$\frac{\partial \bar{A}_\alpha}{\partial \bar{x}^\beta} = \frac{\partial x^\lambda}{\partial \bar{x}^\alpha}\frac{\partial A_\lambda}{\partial \bar{x}^\beta} + \frac{\partial^2 x^\lambda}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} A_\lambda$$

$$= \frac{\partial x^\lambda}{\partial \bar{x}^\alpha}\frac{\partial A_\lambda}{\partial x^\mu}\frac{\partial x^\mu}{\partial \bar{x}^\beta} + \frac{\partial^2 x^\lambda}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} A_\lambda$$

$$= \frac{\partial x^\lambda}{\partial \bar{x}^\alpha}\frac{\partial x^\mu}{\partial \bar{x}^\beta}\frac{\partial A_\lambda}{\partial x^\mu} + \frac{\partial^2 x^\lambda}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} A_\lambda \qquad \text{(ii)}$$

The presence of second term in equation (ii) shows that $\frac{\partial A_\lambda}{\partial x_\mu}$ does not transform as tensor laws. Hence $\frac{\partial A_\lambda}{\partial x_\mu}$ is not a tensor.

Q. If A^i and B_j are the components of a Contravariant and covariant tensors of rank one, then show that $C^i_j = A^i B_j$ are the components of a mixed tensors of rank two.

We have

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^\alpha} A^\alpha$$

and $$\bar{B}_j = \frac{\partial x^\beta}{\partial \bar{x}^j} B_\beta$$

So that; $$\bar{C}^i_j = \bar{A}^i \bar{B}_j = A^\alpha B_\beta \frac{\partial \bar{x}^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \bar{x}^j} \qquad \text{(i)}$$

Equation (i) shows that C^i_j transforms as a mixed tensor of rank two.

Q. If xy, $2y\text{-}z^2$ and xz are the components of a covariant tensor in rectangular coordinates, then find its covariant components in spherical coordinates.

Suppose $x = x_1$, $y = x_2$ and $z = x_3$; then the covariant components of the tensor A_λ are;

$A_1 = xy = x_1 x_2$

$A_2 = 2y - z^2 = 2x_2 - x_3^2$

$A_3 = xz = x_1 x_3$

Let $\bar{A}_\mu$ be the covariant components in spherical coordinates.

Then, $\bar{x}_1 = r, \bar{x}_2 = \theta, \bar{x}_3 = \varphi$ and $\bar{A}_\mu = \dfrac{\partial x^\lambda}{\partial \bar{x}^\mu} A_\lambda$ (i)

Now the transformation equations are,

$x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$

In the existing case;

$x_1 = \bar{x}_1 \sin\bar{x}_2 \cos\bar{x}_3$, $x_2 = \bar{x}_1 \sin\bar{x}_2 \sin\bar{x}_3$, $x_3 = \bar{x}_1 c\cos\bar{x}_2$

Now from equation (i):

$$\bar{A}_1 = \frac{\partial x_1}{\partial \bar{x}_1} A_1 + \frac{\partial x_2}{\partial \bar{x}_1} A_2 + \frac{\partial x_3}{\partial \bar{x}_1} A_3$$

$$= (\sin\bar{x}_2 \cos\bar{x}_3) x_1 x_2 + \sin\bar{x}_2 \sin\bar{x}_3 (2x_2 - x_3^2) + \cos\bar{x}_2 . x_1 x_3$$

$$= (\sin\theta\cos\phi)(r^2 \sin^2\theta \sin\phi\cos\phi) + \sin\theta\sin\phi(2r\sin\theta\sin\phi - r^2\cos^2\theta)$$

$$+ \cos\theta(r^2 \sin\theta\cos\theta\cos\phi)$$

Similarly:

$$\bar{A}_2 = (r\cos\theta\cos\phi)(r^2\sin^2\theta\sin\phi\cos\phi) + (r\cos\theta\sin\phi)(2r\sin\theta\sin\phi - r^2\cos^2\theta)$$

$$+(-r\sin\theta(r^2\sin\theta\cos\theta\cos\phi)$$

$$\bar{A}_3 = (-r\sin\theta\sin\phi)(r^2\sin^2\theta\sin\phi\cos\phi) + (r\sin\theta\cos\phi)(2r\sin\theta\sin\phi - r^2\cos^2\theta)$$

Q. Find the components of a vector in polar co-ordinate whose components in Cartesian coordinates are $\dot{x}, \dot{y}$ and $\ddot{x}, \ddot{y}$.

Levi Civita tensor:

Levi Civita symbol ε^{ijk} or ε_{ijk} is a invariant component tensor, which is antisymmetric in its all indices.

$\therefore +\varepsilon_{ijk} = -\varepsilon_{jik} = +\varepsilon_{jki} = -\varepsilon_{ikj} = +\varepsilon_{kij} = -\varepsilon_{kji}$

But if I = j

$\therefore +\varepsilon_{ijk} = -\varepsilon_{jik}$

$\Rightarrow \varepsilon_{iik} = -\varepsilon_{iik}$

$\Rightarrow \varepsilon_{iik} = 0$

Thus it is clear that whenever two indices are equal the component is zero. Thus although a tensor of 3 rd rank in 3 dimension has 27 components but among them only 6 are non-vanishing. All of them have either +1 or -1.

∴ The only non-vanishing components of ε_{ijk} are those for which all the indices are different and they are equal to +1 or -1 according to as (ijk) is an even or odd permutation of (123) i.e.

$$\varepsilon_{ijk} = \begin{cases} +1 \text{ if (ijk) is an even permutation of (123)} \\ -1 \text{ if (ijk) is an odd permutation of (123)} \\ 0 \text{ otherwise (contain two or more repeated indices)} \end{cases}$$

$$\therefore \varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = +1$$

$$\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1$$

A tensor density is defined to be the quantity which transforms as;

$$\bar{T}^{i_1 i_2 \ldots}_{j_1 j_2 \ldots} = \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}} \frac{\partial \bar{x}^{i_2}}{\partial x^{k_2}} \ldots \frac{\partial \bar{x}^{l_1}}{\partial x^{j_1}} \frac{\partial \bar{x}^{l_2}}{\partial x^{j_2}} \ldots \left| \frac{\partial x^a}{\partial \bar{x}^b} \right|^w T^{k_1 k_2 \ldots}_{l_1 l_2 \ldots}$$

Where w is a constant, the value of which characterizes a tensor and is called the weight of the tensor density.

The Levi Civita symbol ε_{ijk} or ε^{ijk} are the densities of weight -1 or +1.

The Levi Civita symbol ε can be used to write the cross product of two vectors.

Let

$$D = A \times B$$

$$\therefore D_1 = A_2 B_3 - A_3 B_2$$

$$= \varepsilon_{123} A_2 B_3 + \varepsilon_{132} A_3 B_2$$

$$= \varepsilon_{1ij} A_i B_j$$

Similarly;

$$D_2 = \varepsilon_{2ij} A_i B_j$$

$$D_3 = \varepsilon_{3ij} A_i B_j$$

and $\boxed{D_\mu = \varepsilon_{\mu ij} A_i B_j}$

Pseudo Tensor:

If there be a tensor ε_{ijkl} of rank 4, defined such that;

$$\varepsilon_{ijkl} = \begin{cases} +1 \text{ if (ijkl) is an even permutation of (0123)} \\ -1 \text{ if (ijkl) is an odd permutation of (0123)} \\ 0 \text{ otherwise (contain two or more repeated indices)} \end{cases}$$

These are termed as components of **pseudo tensor** of rank 4.

From every antisymmetric tensor $A_{\mu\sigma}$ of the second rank a pseudo tensor $A^*_{\mu\sigma}$ of the same rank can be obtained by multiplying the former with a pseudo tensor of rank 4.

$$\text{i.e. } A^*_{\mu\sigma} = \frac{1}{2}\sum_{\alpha,\beta=0}^{3} \varepsilon^{\mu\sigma\tau\rho} A_{\alpha\beta}$$

Thus the product of a tensor with a pseudo tensor is a pseudo tensor. It is called dual of a given tensor.

Dual of a covariant tensor:

A dual of a covariant tensor $A_{i_1 i_2 \ldots i_n}$ of rank 'n' is defined as the fully Contravariant tensor of rank m – n such that;

$$B^{j_1 j_2 \ldots j_{m-n}} = \varepsilon^{j_1 j_2 \ldots j_{m-n} i_1 i_2 \ldots i_n} A_{i_1 i_2 \ldots i_n}$$

For example A_{ij} is a tensor of rank 2 then its dual of rank one is given by:

$$B^k = \varepsilon^{kij} A_{ij}$$

$$\therefore B^1 = \varepsilon^{123} A_{23} + \varepsilon^{132} A_{32} = A_{23} - A_{32}$$

If A_{ij} is a product of two vectors say $A_{ij} = X_i Y_j$, then we see that the dual of A_{ij} is actually the vector product of the two constituent vectors X_i and Y_j.

Dyads and Dyadic:

Another important representation of a rank 2 tensor is a dyadic. A dyad is a pair of vectors written in a definite order $(\vec{A}.\vec{B})$. The first is known as the antecedent and the second one is the consequent.

The scalar product of a dyadic is a vector but the product can be defined in two ways:

$$(\overrightarrow{A.B}).\vec{C} = \vec{A}(\overrightarrow{B.C})$$

$$\vec{C}.(\overrightarrow{A.B}) = \vec{B}(\overrightarrow{A.C})$$

In the first case $\vec{C}$ is called the postfactor and in the second case, the prefactor. In general the two products are not equal. However the magnitude and direction of the product is generally different from that of $\vec{C}$.

A double dot product of two dyads is defined as;

$$(\overrightarrow{A.C}).(\overrightarrow{B.D}) = \vec{C}(\overrightarrow{A.B}).\vec{D}$$

Dyadic is a linear polynomial of dyads.

$$\text{i.e., } \vec{A}.\vec{B} + \vec{C}\vec{D} + - - -$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\therefore \vec{A}.\vec{B} = A_x B_x \hat{i}\hat{i} + A_x B_y \hat{i}\hat{j} + A_x B_z \hat{i}\hat{k} + A_y B_x \hat{j}\hat{i} + A_y B_y \hat{j}\hat{j} + A_y B_z \hat{j}\hat{k} + A_z B_x \hat{k}\hat{i} + A_z B_y \hat{k}\hat{j} + A_z B_z \hat{k}\hat{k}$$

This is called the nonion form of the dyad because there are nine coefficients.
The unit dyadic is defined as;

$$\hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k} = 1$$

Therefore we have; $1.\vec{A} + \vec{A}.1 = \vec{A}$.

$$A_{11} \quad A_{12} \quad A_{13} \ldots\ldots\ldots\ldots A_{1N}$$
$$A_{21} \quad A_{22} \quad A_{23} \ldots\ldots\ldots\ldots A_{2N}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_{N1} \quad A_{N2} \quad A_{N3} \ldots\ldots\ldots\ldots A_{NN}$$

There are N independent components of the form

$$A_{11}, A_{22}, A_{33}, \ldots\ldots\ldots\ldots A_{NN}$$

Hence no. of component of the form $A_{12}, A_{23}, A_{34}, \ldots\ldots\ldots\ldots$ in which there are distinct subscripts will be $N^2 - N$. But these components are symmetric. *i.e.*, $A_{12} = A_{21}$ etc.

Hence no. of different component of this form are $\frac{1}{2}(N^2 - N)$

$\therefore$ Total no. of different component are

$$= \frac{1}{2}(N^2 - N) + N$$

$$= \frac{N^2}{2} + \frac{N}{2} = \frac{N(N+1)}{2}.$$

Example 15. *Define a metric or fundamental tensor. Determine the components of the fundamental tensor in cylindrical co-ordinates.*

Solution. Metric or Fundamental Tensor.

In rectangular coordinates (x, y, z) the differential of arc length ds is obtained from $ds^2 = dx^2 + dy^2 + dz^2$. By transforming to general curvilinear co-ordinates this becomes

$$ds^2 = \sum_{p=1}^{3} \sum_{q=1}^{3} g_{pq}\, du_p\, du_q$$

Such spaces are called three-dimensional Euclidean spaces. We define the line element ds in this space to be given by the quadratic form, called the metric form or metric,

$$ds^2 = \sum_{p=1}^{N} \sum_{q=1}^{N} g_{pq}\, dx^p\, dx^q$$

or

$$ds^2 = g_{pq}\, dx^p\, dx^q$$

The quantities g_{pq} are the components of a covariant tensor of rank 2 called the *metric* ensor or *fundamental* tensor.

We know that $ds^2 = dx^2 + dy^2 + dz^2$

In cylindrical coordinates,

$$x = r\cos\theta, y = r\sin\theta, z = z$$

$\therefore$ $dx = -r\sin\theta\, d\theta + \cos\theta\, dr$

$$dy = r\cos\theta\, d\theta + \sin\theta\, dr$$

$$dz = dz$$

$\therefore$ $ds^2 = dx^2 + dy^2 + dz^2$

$$= (-r\sin\theta\, d\theta + \cos\theta\, dr)^2 + (r\cos\theta\, d\theta + \sin\theta\, dr)^2 + (dz)^2$$

$$ds^2 = (dr)^2 + r^2 (d\theta)^2 + (dz)^2$$

Also metric is given by

$$ds^2 = g_{pq}\, dx^p\, dx^q \qquad \ldots(2)$$

If $x^1 = r, x^2 = \theta, x^3 = z$

then comparing (1) & (2), we have

$$g_{11} = 1,\ g_{22} = r^2,\ g_{33} = 1,\ g_{12} = g_{21} = 0,\ g_{13} = g_{31} = 0,\ g_{23} = g_{32} = 0.$$

Metric tensor is given by $\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$

Metric tensor in cylindrical co-ordinates $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Ans.

Example 16. *Define what is meant by invariant? Show that the contraction of the outer product of the tensors A^p and B^q is an invariant.*

Solution. Scalar or Invariant. Suppose ϕ is a function of the co-ordinates x^k, and let $\bar{\phi}$ denote the functional value under a transformation to a new set of co-ordinates $\bar{x}^k$. Then ϕ is called a *scalar* or *invariant* with respect to the co-ordinate transfomation if $\phi = \bar{\phi}$.

Since A^p and B_q are tensors.

$$\bar{A}^j = \frac{\partial \bar{x}^j}{\partial x^p} A^p,\ \bar{B}_k = \frac{\partial x^q}{\partial \bar{x}^k} B_q$$

Then

$$\bar{A}^j \bar{B}_k = \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial x^q}{\partial x^{-k}} A^p B_q$$

By contraction (putting $j = k$)

$$\bar{A}^j \bar{B}_j = \frac{\partial \bar{x}^j}{\partial x^p} \frac{\partial x^q}{\partial \bar{x}^j} A^p B_q$$

$$= \delta^q_p A^p B_q = A^p B_p$$

and so $A^p B_p$ is an invariant.

Example 17. *What do you understand by associated tensors ?*

Solution. Associated Tensors. Given a tensor we can derive other tensors by raising or lowering indices. For example, given the tensor A_{pq} we obtain by raising the index p, the tensor A^p_q, the dot indicating the original position of the moved index. By raising the index q also we obtain A^{pq}. We shall often write A^{pq}. These derived tensors can be obtained by forming inner products of the given tensor with the metric tensor g_{pq} or its conjugate g^{pq}. Thus, for example

$$A^p_q = g^{rp} A_{rq},\ , A^{pq} = g^{rp} g^{sq} A_{rs}$$

All tensors obtained from a given tensor by forming inner products with the metric tensor and its conjugate are called *associated* tensors of the given tensor. For example A^m and A_m are associated tensors, the first are contravariant and the second covariant components. The relation between them is given by

$$A_p = g_{pq} A^q \text{ or } A^p = g^{pq} A_q.$$

For rectangular co-ordinates $g_{pq} = 1$ if $p = q$, and 0 if $p \neq q$, so that $A_p = A^p$, which explains why no distinction was made between contravariant and covariant components of a vector for rectangular co-ordinates.

8.42 Conjugate or Reciprocal Tensors

Let $g = |g_{pq}|$ denote the determinant with elements g_{pq} and suppose $g \neq 0$. Define g^{pq} by

$$g^{pq} = \frac{\text{cofactor of } g_{pq}}{g}$$

Then g^{pq} is a symmetric contravariant tensor of rank two called the conjugate or Reciprocal tensor of g_{pq}.

Also $$g^{pq} g_{rq} = \delta^p_r$$

8.43 Christoffel Symbols

The symbols

$$[pq, r] = \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^q} + \frac{\partial g_{qr}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^r} \right), \quad \left\{ \begin{matrix} s \\ pq \end{matrix} \right\} = g^{sr} [pq, r]$$

are called the Christoffel symbols of the first and second kind respectively.

Example 18. *Prove that* $[pq, r] = g_{rs} \left\{ \begin{matrix} s \\ pq \end{matrix} \right\}$.

Solution. $$g_{ks} \left\{ \begin{matrix} s \\ pq \end{matrix} \right\} = g_{ks} g^{sr} [pq, r]$$

$$= \delta^r_k [pq, r]$$

$$= [pq, k]$$

$$[pq, k] = g_{ks} \left\{ \begin{matrix} s \\ pq \end{matrix} \right\}$$

$$[pq, r] = g_{rs} \left\{ \begin{matrix} s \\ pq \end{matrix} \right\}$$

Example 19. *Prove that* (*i*) $[pq, r] = [qp, r]$ (*ii*) $\left\{ \begin{matrix} s \\ pq \end{matrix} \right\} = \left\{ \begin{matrix} s \\ qp \end{matrix} \right\}$

Solution. (*i*) $$[pq, r] = \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^q} + \frac{\partial g_{qr}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^r} \right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{qr}}{\partial x^p} + \frac{\partial g_{pr}}{\partial x^q} - \frac{\partial g_{qp}}{\partial x^r} \right)$$

$$[pq, r] = [qp, r]$$

(*ii*) $$\left\{ \begin{matrix} s \\ pq \end{matrix} \right\} = g^{sr} [pq, r]$$

$$= g^{sr} [qp, r] = \left\{ \begin{matrix} s \\ qp \end{matrix} \right\}$$

Example 20. *Prove* $\dfrac{\partial g_{pq}}{\partial x^m} = [pm, q] + [qm, p]$

Solution. $[pm, q] + [qm, p]$

$$= \frac{1}{2}\left(\frac{\partial g_{pq}}{\partial x^m} + \frac{\partial g_{mq}}{\partial x^p} - \frac{\partial g_{pm}}{\partial x^q} \right) + \frac{1}{2}\left(\frac{\partial g_{qp}}{\partial x^m} + \frac{\partial g_{mp}}{\partial x^q} - \frac{\partial g_{qm}}{\partial x^p} \right)$$

$$= \frac{1}{2}\frac{\partial g_{pq}}{\partial x^m} + \frac{1}{2}\frac{\partial g_{qp}}{\partial x^m} = \frac{\partial g_{pq}}{\partial x^m}$$

Example 21. $\dfrac{\partial g^{pq}}{\partial x^m} = -g^{pn}\begin{Bmatrix} q \\ mn \end{Bmatrix} - g^{qn}\begin{Bmatrix} p \\ mn \end{Bmatrix}$

Solution. $\dfrac{\partial}{\partial x^m}\left(g^{jk}\, g_{ij}\right) = \dfrac{\partial}{\partial x^m}\left(\delta_i^k\right) = 0$

Then $$g^{jk}\frac{\partial g_{ij}}{\partial x^m} + \frac{\partial g^{jk}}{\partial x^m} g_{ij} = 0$$

or $$g_{ij}\frac{\partial g^{jk}}{\partial x^m} = -g^{jk}\frac{\partial g_{ij}}{\partial x^m}$$

Multiplying by g^{ir}

i.e. $$\delta_j^r \frac{\partial g^{jk}}{\partial x^m} = -g^{ir}\, g^{jk}\,[im, j] + [jm, i]$$

or $$\frac{\partial g^{rk}}{\partial x^m} = -g^{ir}\begin{Bmatrix} k \\ im \end{Bmatrix} - g^{jk}\begin{Bmatrix} r \\ jm \end{Bmatrix}$$

Replace r, k, i, j by p, q, n, n

$$\frac{\partial g^{pq}}{\partial x^m} = -g^{pn}\begin{Bmatrix} q \\ mn \end{Bmatrix} - g^{qn}\begin{Bmatrix} p \\ mn \end{Bmatrix}$$

Example 22. *Derive transformation laws for the christoffel symbols of the first and second kind.*

Solution. Since $\bar{g}_{jk} = \dfrac{\partial x^p}{\partial \bar{x}^j}\dfrac{\partial x^q}{\partial \bar{x}^k} g_{pq}$

$$\therefore \quad \frac{\partial \bar{g}_{jk}}{\partial \bar{x}^m} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial g_{pq}}{\partial x^r}\frac{\partial x^r}{\partial \bar{x}^m} + \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial^2 x^q}{\partial \bar{x}^m\,\partial \bar{x}^k} g_{pq} + \frac{\partial^2 x^p}{\partial \bar{x}^m\,\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k} g_{pq} \quad \ldots(1)$$

By cyclic permutation of indices n, k, m and p, q, r

$$\frac{\partial \bar{g}_{km}}{\partial \bar{x}^j} = \frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m}\frac{\partial g_{qr}}{\partial x^p}\frac{\partial x^p}{\partial \bar{x}^j} + \frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial^2 x^r}{\partial \bar{x}^j\,\partial \bar{x}^m} g_{qr} + \frac{\partial^2 x^q}{\partial \bar{x}^j\,\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m} g_{qr} \quad \ldots(2)$$

$$\frac{\partial \bar{g}_{mj}}{\partial \bar{x}^k} = \frac{\partial \bar{x}^r}{\partial \bar{x}^m}\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial g_{rp}}{\partial x^q}\frac{\partial x^q}{\partial \bar{x}^k} + \frac{\partial x^r}{\partial \bar{x}^m}\frac{\partial^2 x^p}{\partial \bar{x}^k\,\partial \bar{x}^j} g_{rp} + \frac{\partial^2 x^r}{\partial \bar{x}^k\,\partial \bar{x}^m}\frac{\partial x^p}{\partial \bar{x}^j} g_{rp} \quad \ldots(3)$$

Subtracting (1) from the sum of (2) and (3) and multiplying by $\frac{1}{2}$, we obtain on using the definition of the Christoffel symbols of the first kind,

$$[\overline{jk, m}] = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m}[pq, r] + \frac{\partial^2 x^p}{\partial \bar{x}^j\,\partial \bar{x}^k}\frac{\partial x^q}{\partial \bar{x}^m} g_{pq} \quad \ldots(4)$$

18.44 Transformation law for second kind

Multiplying (4) by $\bar{g}^{nm}$

$$\bar{g}^{nm} = \frac{\partial \bar{x}^n}{\partial x^s}\frac{\partial \bar{x}^m}{\partial x^t} g^{st} \quad \text{we get}$$

$$\bar{g}^{nm}[\overline{jk, m}] = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial x^r}{\partial \bar{x}^m}\frac{\partial \bar{x}^n}{\partial x^s}\frac{\partial \bar{x}^m}{\partial x^t} g^{st}[pq, r] + \frac{\partial^2 x^p}{\partial \bar{x}^j\,\partial \bar{x}^k}\frac{\partial x^q}{\partial \bar{x}^m}\frac{\partial \bar{x}^n}{\partial x^s}\frac{\partial \bar{x}^m}{\partial x^t} g^{st}$$

Then
$$\overline{\begin{Bmatrix} n \\ jk \end{Bmatrix}} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial \bar{x}^n}{\partial x^r}\delta^l_i\, g^{ri}\,[pq, r] + \frac{\partial^2 x^p}{\partial \bar{x}^j\,\partial \bar{x}^k}\frac{\partial \bar{x}^n}{\partial x^r}\delta^l_i\, g^{ri}\, g_{pq}$$

$$\overline{\begin{Bmatrix} n \\ jk \end{Bmatrix}} = \frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}\frac{\partial \bar{x}^n}{\partial x^r}\begin{Bmatrix} s \\ pq \end{Bmatrix} + \frac{\partial^2 x^p}{\partial \bar{x}^j\,\partial \bar{x}^k}\frac{\partial \bar{x}^n}{\partial x^p} \qquad (5)$$

(4) and (5) are required transformation laws.

Example 23. *If* $(ds)^2 = r^2\,(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2$, *find the value of*

(*a*) [22, 1] (*b*) [12, 2] (*c*) [1, 22] (*d*) [2, 12].

Solution. $(ds)^2 = r^2\,(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2$

$$g_{11} = r^2, \quad g_{22} = r^2\sin^2\theta, \quad g_{12} = 0 = g_{21}$$

$$g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = \begin{vmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix} = r^4\sin^2\theta$$

$$g^{11} = \frac{\text{cofactor of } g_{11}}{g} = \frac{r^2\sin^2\theta}{r^4\sin^2\theta} = \frac{1}{r^2}$$

$$g^{22} = \frac{\text{cofactor of } g_{22}}{g} = \frac{r^2}{r^4\sin^2\theta} = \frac{1}{r^2\sin^2\theta}$$

The christoffel symbols of first kind are

$$[ij, k] = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right]$$

(*a*)
$$[22, 1] = \frac{1}{2}\left[\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right]$$

$$= \frac{1}{2}\left[\frac{\partial(0)}{\partial\phi} + \frac{\partial(0)}{\partial\phi} - \frac{\partial(r^2\sin^2\theta)}{\partial\theta}\right]$$

$$= r^2\sin\theta\cos\theta$$

(*b*)
$$[12, 2] = \frac{1}{2}\left[\frac{\partial g_{22}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^2}\right]$$

$$= \frac{1}{2}\left[\frac{\partial(r^2\sin^2\theta)}{\partial\theta} + \frac{\partial(0)}{\partial\phi} - \frac{\partial(0)}{\partial\phi}\right]$$

$$= r^2\sin\theta\cos\theta$$

(*c*) The christoffel symbols of second kind are

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = g^{kl}\,[ij, l]$$

$$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = g^{1l}\,[22, l]$$

$$= g^{11}\,[22, 1] + g^{12}\,[22, 2]$$

$$= \frac{1}{r^2}[-r^2\sin\theta\cos\theta] + 0 \qquad (g^{12} = 0)$$

$$= -\sin\theta\cos\theta$$

(d) $$\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = g^{2l}[12, l]$$

$$= g^{21}[12, 1] + g^{22}[12, 2]$$

$$= 0 + \frac{1}{r^2\sin^2\theta}[r^2\sin\theta\cos\theta] = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

$$= r^4\sin\theta\cos\theta$$ Ans.

Example 24. *If* $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2(d\phi)^2$, *find the value of*

(a) $[22, 1]$, (b) $[33, 1]$, (c) $[13, 3]$, (d) $[23, 3]$,

(e) $\begin{Bmatrix} 1 \\ 22 \end{Bmatrix}$, (f) $\begin{Bmatrix} 1 \\ 33 \end{Bmatrix}$, (g) $\begin{Bmatrix} 3 \\ 13 \end{Bmatrix}$, (h) $\begin{Bmatrix} 3 \\ 23 \end{Bmatrix}$.

Solution. $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2(d\phi)^2$

$$x_1 = r, \quad x_2 = \theta, \quad x_3 = \phi$$

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2\sin\theta$$

$$g_{12} = 0, \quad = g_{13} = \ldots\ldots$$

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{vmatrix} = r^4\sin^2\theta$$

$$g^{11} = \frac{\text{cofactor of } g_{11}}{g} = \frac{\begin{vmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix}}{r^4\sin^2\theta} = \frac{r^4\sin^2\theta}{r^4\sin^2\theta} = 1$$

$$g^{22} = \frac{\text{cofactor of } g_{22}}{g} = \frac{\begin{vmatrix} 1 & 0 \\ 0 & r^2\sin^2\theta \end{vmatrix}}{r^4\sin^2\theta} = \frac{r^2\sin^2\theta}{r^4\sin^2\theta} = \frac{1}{r^2}$$

$$g^{33} = \frac{\text{cofactor of } g_{33}}{g} = \frac{\begin{vmatrix} 1 & 0 \\ 0 & r^2 \end{vmatrix}}{r^4\sin^2\theta} = \frac{r^2}{r^4\sin^2\theta} = \frac{1}{r^2\sin^2\theta}$$

The christoffel symbols of the first kind are

$$[ij, k] = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right]$$

(a) $$[22, 1] = \frac{1}{2}\left[\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right]$$

$$= \frac{1}{2}\left[\frac{\partial(0)}{\partial\theta} + \frac{\partial(0)}{\partial\theta} - \frac{\partial(r^2)}{\partial r}\right] = \frac{1}{2}(-2r) = -r$$

(b) $$[33, 1] = \frac{1}{2}\left[\frac{\partial g_{31}}{\partial x^3} + \frac{\partial g_{31}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^1}\right]$$

$$= \frac{1}{2}\left[\frac{\partial(0)}{\partial\phi} + \frac{\partial(0)}{\partial\phi} - \frac{\partial(r^2\sin^2\theta)}{\partial r}\right] = \frac{1}{2}(-2r\sin^2\theta) = -r\sin^2\theta$$

(c) $$[13, 3] = \frac{1}{2}\left[\frac{\partial g_{33}}{\partial x^1} + \frac{\partial g_{13}}{\partial x^3} - \frac{\partial g_{13}}{\partial x^3}\right] = \frac{1}{2}\frac{\partial g_{33}}{\partial x^1}$$

$$= \frac{1}{2}\left[\frac{\partial (r^2 \sin^2 \theta)}{\partial r}\right] = r \sin^2 \theta$$

(d) $$[23, 3] = \frac{1}{2}\left[\frac{\partial g_{33}}{\partial x^2} + \frac{\partial g_{23}}{\partial x^3} - \frac{\partial g_{23}}{\partial x^3}\right] = \frac{1}{2}\frac{\partial g_{33}}{\partial x^2}$$

$$= \frac{1}{2}\frac{\partial}{\partial \theta}(r^2 \sin^2 \theta) = r^2 \sin \theta \cos \theta$$

The christoffel symbols of the second kind are

$$\left\{ \begin{matrix} k \\ ij \end{matrix} \right\} = g^{kl}[ij, l]$$

(e) $$\left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} = g^{1l}[22, l]$$

$$= g^{11}[22, 1] + g^{12}[22, 2] + g^{13}[22, 3]$$

$$= (1)(-r) + 0 + 0 = -r$$

(f) $$\left\{ \begin{matrix} 1 \\ 33 \end{matrix} \right\} = g^{1l}[33, l]$$

$$= g^{11}[33, 1] + g^{12}[33, 2] + g^{13}[33, 3]$$

$$= (1)(-r \sin^2 \theta) + 0 + 0 = -r \sin^2 \theta$$

(g) $$\left\{ \begin{matrix} 3 \\ 13 \end{matrix} \right\} = g^{3l}[13, l]$$

$$= g^{31}[13, 1] + g^{32}[13, 2] + g^{33}[13, 3]$$

$$= 0\,(r \sin^2 \theta) + 0\,[0] + \frac{1}{r^2 \sin^2 \theta}[r \sin^2 \theta] = \frac{1}{r}$$

(h) $$\left\{ \begin{matrix} 3 \\ 23 \end{matrix} \right\} = g^{3l}[23, l]$$

$$= g^{31}[23, 1] + g^{32}[23, 2] + g^{33}[23, 3]$$

$$= 0\,[0] + 0\,[0] + \frac{1}{r^2 \sin^2 \theta}(r \sin \theta \cos \theta) = \cot \theta$$

Example 25. *Prove that* $\left\{ \begin{matrix} p \\ pq \end{matrix} \right\} = \frac{\partial}{\partial x^q} \log \sqrt{g}$

Solution. $g = g_{jk}\,G(j, k)$(Sum over k only)

where $G(j, k)$ is the cofactor of g_{jk} in the determinant $g = |g_{jk}| \neq 0$ since $G(j, k)$ does not contain g_{jk} explicitly,

$$\frac{\partial g}{\partial g_{jr}} = G(j, r)$$

Then, summing over j and r

$$\frac{\partial g}{\partial x^m} = \frac{\partial g}{\partial g_{jr}}\frac{\partial g_{jr}}{\partial x^m} = G(j, r)\frac{\partial g_{jr}}{\partial x^m}$$

$$= g\, g^{pr} \frac{\partial g_{jr}}{\partial x^m}$$

$$= g\, g^{pr} [jm, r] + [rm, j])$$

$$\frac{\partial g}{\partial x^m} = g\left(\begin{Bmatrix} j \\ jm \end{Bmatrix} + \begin{Bmatrix} r \\ rm \end{Bmatrix} \right) = 2g \begin{Bmatrix} j \\ jm \end{Bmatrix}$$

Thus
$$\frac{1}{2g} \frac{\partial g}{\partial x^m} = \begin{Bmatrix} j \\ jm \end{Bmatrix}$$

or
$$\begin{Bmatrix} j \\ jm \end{Bmatrix} = \frac{\partial}{\partial x^m} \log \sqrt{g}$$

Replacing j by p and m by q,

$$\begin{Bmatrix} p \\ pq \end{Bmatrix} = \frac{\partial}{\partial x^q} \log \sqrt{g}$$

11.45 Contravariant, Covariant and Mixed Tensor

If A_i be a set of n functions of the co-ordinates $x^1, x^2, x^3 \ldots x^n$ (x^i). They are transformed to another system of co-ordinates $\bar{x}^1, \bar{x}^2, \bar{x}^3 \ldots \bar{x}^n$ according to $\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j$

A_i are called the components of a covariant tensor.

If $\phi(x^1, x^2, \ldots, x^n)$ be a scalar functions, then

$$\frac{\partial \phi}{\partial \bar{x}^i} = \frac{\partial \phi}{\partial x^1} \frac{\partial x^1}{\partial \bar{x}^i} + \frac{\partial \phi}{\partial x^2} \frac{\partial x^2}{\partial \bar{x}^i} + \ldots + \frac{\partial \phi}{\partial x^n} \frac{\partial x^n}{\partial \bar{x}^i}$$

then $\frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2} \ldots \frac{\partial \phi}{\partial x^n}$ are the components of a covariant vector.

Since $\bar{x}$ is a function of x^i (i.e., $x^1, x^2, \ldots x^n$)

so
$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^1} dx^1 + \frac{\partial \bar{x}^i}{\partial x^2} dx^2 + \ldots + \frac{\partial \bar{x}^i}{\partial x^n} dx^n$$

$$= \frac{\partial \bar{x}^i}{\partial x^j} dx^j$$

On comparing (1) and (2) we can say that $dx^1, dx^2 \ldots dx^n$ is an example of a contravariant tensor.

If q and s varry from 1 to n, then A^{qs} will be n^2 functions.

If N^2 quantities A^{qs} in a co-ordinate system $(x^1, x^2, \ldots x^N)$ are related to N^2 other quantities $\bar{A}^{pr}$ in another system $(\bar{x}^1, \bar{x}^2, \ldots \bar{x}^N)$ by the transformation equations

$$\bar{A}^{pr} = \sum_{s=1}^{N} \sum_{q=1}^{N} \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs}$$

or
$$\bar{A}^{pr} = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial \bar{x}^r}{\partial x^s} A^{qs},$$

they are called *contravariant* components of a tensor of the second rank.

If the transformation law is

$$\bar{A}_{pr} = \frac{\partial x^q}{\partial \bar{x}^p} \frac{\partial x^s}{\partial \bar{x}^r} A_{qs},$$

then quantities A_{qs} are called components of *covariant* tensor of second rank.

The N^2 quantities A^q_s are called components of a *mixed* tensor of second rank if

$$\bar{A}^p_r = \frac{\partial \bar{x}^p}{\partial x^q} \frac{\partial x^s}{\partial \bar{x}^r} A^q_s.$$

EXERCISE 18.2

1. If A^i are the components of an absolute contravariant tensor of rank one, show that $\frac{\partial A_i}{\partial x_j}$ are the components of a mixed tensor.

2. If A^{ij} and A_{ij} are reciprocal symmetric tensors and x_i are the components of a covariant tensor of rank one, show that $A_{ij} x^i x^j = A^{ij} x_i x_j$, where $x^i = A^{i\alpha} x_\alpha$.

3. If the components of a tensor are zero in one co-ordinate system, then prove that the components are zero in all co-ordinate systems.

4. Show that the expression $A(i, j, k)$ is a tensor if its inner product with an arbitrary tensor B^j_k is a tensor.

5. A^{ij} is a contravariant tensor and B_i a covariant tensor. Show that $A^{ij} B_k$ is a tensor of rank three, but $A^{ij} B_j$ is a tensor of rank one.

6. If g_{ij} denotes the components of a covariant tensor of rank two, show that the product $g_{ij}\, dx^i\, dx^j$ is an invariant scalar.

7. Find g and g^{ij} corresponding to the metric

$$ds^2 = 5\,(dx^1)^2 + 3\,(dx^2)^2 + 4\,(dx^3)^2 - 6\,dx^1\,dx^2 + 4dx^2\,dx^3.$$

Ans. $g = 4, g^{11} = 2, g^{22} = 5, g^{33} = 1.5, g^{12} = 3, g^{23} = -2.5, g^{13} = -1.5$

8. Find the values of g and g^{ij}, if

$$ds^2 = \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) \text{ where } R \text{ is constant.}$$

Ans. $g = \dfrac{r^4 \sin^2\theta}{1 - \frac{r^2}{R^2}}$; $g^{11} = 1 - \dfrac{r^2}{R^2}$, $g^{22} = \dfrac{1}{r^2}$, $g^{33} = \dfrac{1}{r^2 \sin^2\theta}$, $g^{ij} = 0\ (i \neq j)$

9. Prove that the angle $\theta_{12}, \theta_{23}, \theta_{31}$ between the co-ordinate curves in a three dimensional co-ordinate system are given by

$$\cos\theta_{12} = \frac{g_{12}}{\sqrt{g_{11}\, g_{22}}}, \cos\theta_{23} = \frac{g_{23}}{\sqrt{g_{22}\, g_{33}}}, \cos\theta_{31} = \frac{g_{31}}{\sqrt{g_{33}\, g_{11}}}.$$

10. Prove that for an orthogonal co-ordinate system

(a) $g_{12} = g_{23} = g_{31} = 0$ (b) $g^{11} = \dfrac{1}{g_{11}}, g^{22} = \dfrac{1}{g_{22}}, g^{33} = \dfrac{1}{g_{33}}$

11. Surface of a sphere is a two dimensional Riemannian space. Find its fundamental metric tensor. If a be the fixed radius of the sphere.

Ans. $g_{11} = a^2, g_{22} = a^2 \sin^2\theta, g = a^4 \sin^2\theta$

$$g^{11} = \frac{1}{a^2}, \quad g^{22} = \frac{1}{a^2 \sin^2\theta} \quad g^{12} = 0 = g^{21}.$$

Christoffel symbols

The derivative of a tensor doesnot always lead to another tensor.

Let us consider a tensor of rank one.

$$A^j = \frac{\partial x^j}{\partial \bar{x}^i} \bar{A}^i$$

Now differentiate the above we get,

$$\frac{\partial A^j}{\partial x^k} = \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial \bar{x}^m}{\partial x^k} \frac{\partial \bar{A}^i}{\partial \bar{x}^m} + \frac{\partial^2 x^j}{\partial x^k . \partial \bar{x}^i} \bar{A}^i$$

The second term in the right hand side of the above equation suggests that the derivative of A^j is not a tensor.

The aim of tensor calculus is to develope expressions involving the derivatives of a tensor which are the components of a tensor.

Accordingly Christoffel symbols have been developed/introduced in the tensor analysis.

Christoffel symbol of first and second kind have been defined respectively as —

$$[pq, r] = \frac{1}{2}\left(\frac{\partial g_{pr}}{\partial x^q} + \frac{\partial g_{qr}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^r}\right)$$

$$\left\{ {s \atop pq} \right\} = g^{sr} [pq, r]$$

which are formed from the fundamental tensor g_{pq}.

From the definition it is seen that both the symbols are symmetric w.r.to the indices p and q.

$\therefore$ $[ij, k] = [ji, k]$ and $\left\{ {k \atop ij} \right\} = \left\{ {k \atop ji} \right\}$

Ex-1: Prove that $\varepsilon_{ilm}\varepsilon_{jlm} = 2\delta_{ij}$

Proof: We know that $\varepsilon_{ilm}\varepsilon_{jkm} = \delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj}$

Considering $k = l$ we get,

$$\varepsilon_{ilm}\varepsilon_{jlm} = \delta_{ij}\delta_{ll} - \delta_{il}\delta_{lj}$$

Now, $\delta_{ll} = \delta_{11} + \delta_{22} + \delta_{33} = 1+1+1 = 3$

$$\delta_{il}\delta_{lj} = \delta_{ij} \quad \left[\because \delta_{ij}a_{im} = a_{jm}\right.$$

$\therefore$ $\varepsilon_{ilm}\varepsilon_{jlm} = 3\delta_{ij} - \delta_{ij}$

$$\Rightarrow \boxed{\varepsilon_{ilm}\varepsilon_{jlm} = 2\delta_{ij}}$$

Ex-2: Prove that $\varepsilon_{ijk}\varepsilon_{ijk} = 6$

Proof: We know that $\varepsilon_{ilm}\varepsilon_{jkm} = \delta_{ij}\delta_{lk} - \delta_{ik}\delta_{lj}$

Considering $k = l$ we get

$$\varepsilon_{ilm}\varepsilon_{jlm} = 2\delta_{ij} \quad \left[\text{using problem ex-1 (above)}\right]$$

Now considering $i = j$

$$\varepsilon_{jlm}\varepsilon_{jlm} = 2\delta_{jj} = 2\times 3$$

$$\Rightarrow \boxed{\varepsilon_{jlm}\varepsilon_{jml} = 6}$$

Note: Both of these problems are taken from Mathematical Physics — by H. K. Das.